Palindromic matrix polynomials, matrix functions and integral representations

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Abstract

We study the properties of palindromic quadratic matrix polynomials $\varphi(z) = P + Qz + Pz^2$, i.e., quadratic polynomials where the coefficients P and Q are square matrices, and where the constant and the leading coefficients are equal. We show that, for suitable choices of the matrix coefficients P and Q, it is possible to characterize by means of $\varphi(z)$ well known matrix functions, namely the matrix square root, the matrix polar factor, the matrix sign and the geometric mean of two matrices. Finally we provide some integral representations of these matrix functions.

Keywords Palindromic matrix polynomial, quadratic matrix equation, matrix function, matrix square root, polar decomposition, matrix sign, matrix geometric mean.

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1 Introduction

Consider the function $\varphi(z) : \mathbb{C} \longrightarrow \mathbb{C}^{n\times n}$ defined as

$$\varphi(z) = P + Qz + Pz^2, \tag{1}$$

where P and Q are $n \times n$ complex matrices, and P is different from the null matrix. The function $\varphi(z)$ is a quadratic matrix polynomial, since $\varphi(z)$ is a matrix whose entries are polynomials of degree at most 2; moreover, since P is different from the null matrix, there exists at least one entry of $\varphi(z)$ which is a polynomial of degree 2. Besides being a quadratic matrix polynomial, $\varphi(z)$ is palindromic, i.e., the constant coefficient is equal to the leading coefficient. Therefore we will refer to $\varphi(z)$ as a *quadratic palindromic matrix polynomial* (QPMP). Throughout the paper we assume that the matrix Q is nonsingular.

Our interest in palindromic matrix polynomials concerns their relationship with certain matrix functions. Matrix functions intervene in many applications, ranging from stochastic processes to control theory, computer graphics, medical diagnostics, and more. For a thorough treatise on matrix functions we refer the reader to the book [9].

The specific matrix functions that we study are the principal matrix square root, the matrix sign, the unitary polar factor and the geometric mean of two positive definite matrices; these functions will be denoted by the symbols $A^{1/2}$, $\mathrm{sign}(A)$, $\mathrm{polar}(A)$ and $A\#B$, respectively. We will call *matrix function* also the matrix geometric mean and the unitary polar factor, even though they are not properly matrix functions according to the customary definition (see [9]).

These four matrix functions are deeply related to each other. For instance, $\mathrm{sign}(A)$, $\mathrm{polar}(A)$ and $A\#B$ can be expressed in terms of suitable matrix square roots (see [1, 9]):

$$\begin{aligned}
&\mathrm{sign}(A) = A(A^2)^{-1/2},\\
&\mathrm{polar}(A) = A(A^*A)^{-1/2},\\
&A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.
\end{aligned}$$

Moreover, $A^{1/2}$ and $\mathrm{polar}(A)$ can be expressed by means of the matrix sign (see [9]):

$$\begin{aligned}
&\mathrm{sign}\left(\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & A^{1/2} \\ A^{-1/2} & 0 \end{bmatrix},\\
&\mathrm{sign}\left(\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \mathrm{polar}(A) \\ (\mathrm{polar}(A))^* & 0 \end{bmatrix},\\
&\mathrm{sign}\left(\begin{bmatrix} 0 & A \\ B^{-1} & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & A\#B \\ (A\#B)^{-1} & 0 \end{bmatrix}.
\end{aligned}$$

In this paper we show further and inner relationships among $A^{1/2}$, $\mathrm{sign}(A)$, $\mathrm{polar}(A)$ and $A\#B$. A common feature of the four matrix functions is the strict connection with the quadratic palindromic matrix polynomial (1).

Recently, much attention has been devoted in the literature [3, 12, 13, 15] to theoretical and computational properties of a different kind of palindromic matrix polynomials, namely the $\star$-palindromic matrix polynomials. A matrix polynomial $L(z) = \sum_{j=0}^{\ell} A_j z^j$ is said to be $\star$-palindromic if $A_i = A_{\ell-i}^{\star}$, for $i = 0, \ldots, \ell$, where the symbol "$\star$" denotes Hermitian transposition if the coefficients belong to $\mathbb{C}$, transposition if the coefficients belong to $\mathbb{R}$. The interest in $\star$-palindromic matrix polynomials is mainly addressed to the computation of their roots, i.e., the solutions of the equation $\det(L(z)) = 0$.

Here, we present some new results on QPMP of the form (1). In particular we derive necessary and sufficient conditions for the existence and uniqueness of a solution X_* of the matrix equation

$$P + QX + PX^2 = 0$$

with spectral radius at most one and which is a function of the matrix $M = Q^{-1}P$. In the case of existence and uniqueness we provide an explicit expression of X_* in terms of the coefficients P and Q.

When X_* has spectral radius less than one, the Laurent matrix polynomial

$$\mathcal{L}(z) = Pz^{-1} + Q + Pz$$

is invertible in an annulus containing the unit circle, and its inverse $\mathcal{H}(z) = \mathcal{L}^{-1}(z)$ has a power series expansion of the form

$$\mathcal{H}(z) = H_0 + \sum_{i=1}^{\infty} H_i(z^i + z^{-i}).$$

The constant term H_0 has a special role in our analysis. Indeed, by extending the result of [16], we show that for particular choices of the coefficients P and Q, the matrix H_0 coincides with one of the matrix functions $A^{1/2}$, $\mathrm{sign}(A)$, $\mathrm{polar}(A)$ and $A\#B$. For instance, if $P = \frac{1}{4}(A - B)$ and $Q = \frac{1}{2}(A + B)$, then $H_0 = A\#B$. This relationship with H_0, besides being interesting *per se*, allows one to give new integral representations of the four matrix functions. Finally, we compare the new integral representations of $A^{1/2}$, $\mathrm{sign}(A)$, $\mathrm{polar}(A)$ and $A\#B$ with known integral representations of the same matrix functions. Surprisingly, we find that the new and the known representations can be all derived from the Cauchy integral formula for the principal inverse square root of a certain matrix.

The paper is organized as follows. In Section 2 we recall some preliminary definitions. Section 3 is devoted to the theoretical analysis of QPMPs. Section 4 concerns the relationships among QPMPs and $A^{1/2}$, $\mathrm{sign}(A)$, $\mathrm{polar}(A)$ and $A\#B$: we specialize the results of Section 3 and characterize the four matrix functions in terms of the constant coefficient of $\mathcal{L}(z)^{-1}$. We analyze the integral representations in Section 5.

2 Preliminaries

We give some preliminary definitions and results about matrix functions and matrix equations, which will be useful in the following sections.

In the following, given a complex number $\zeta \in \mathbb{C} \setminus (-\infty, 0)$, we will denote by $\zeta^{1/2}$ the solution of the equation $x^2 = \zeta$ having nonnegative real part. A matrix square root of A is a matrix X satisfying the equation $X^2 = A$. If A has no nonpositive real eigenvalues, then there exists a unique matrix square root whose eigenvalues have positive real part [4, 7]; such matrix square root is called the *principal matrix square root* and is denoted by $A^{1/2}$. If A has no negative real eigenvalues and the eigenvalue 0 is semisimple, i.e., its algebraic and geometric multiplicities coincide, then A has a unique matrix square root whose nonzero eigenvalues have positive real part (compare [9, Exercise 1.24]). We denote it by $A^{1/2}$ and call it principal matrix square root as well.

Given A with no purely imaginary eigenvalues, let

$$J = V^{-1}AV = \begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix}$$

its Jordan decomposition, where the eigenvalues of J_- have negative real part, while the eigenvalues of J_+ have positive real part. The *matrix sign of* A is defined as

$$\mathrm{sign}(A) = V \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix} V^{-1}$$

where p is the size of J_- and q is the size of J_+ (see for instance [8, 9]).

Given a nonsingular matrix A, the *unitary polar factor of* A, denoted by $\mathrm{polar}(A)$, is the unique matrix U such that

$$A = UH, \quad U^*U = I,$$

where H is Hermitian positive definite (see [8, 9]).

Given A and B Hermitian positive definite matrices, the *matrix geometric mean* of A and B is defined as

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2},$$

or, equivalently, $A\#B = A(A^{-1}B)^{1/2}$ (see for instance [1]).

Given a matrix $A \in \mathbb{C}^{n\times n}$ and a (possibly complex valued) function f, which is sufficiently regular, it is possible to define the matrix function $f(A)$. Here we recall a definition of $f(A)$ and some important properties.

A definition of $f(A)$ can be given in terms of the Jordan canonical form of A, say $M^{-1}AM = J_1 \oplus \cdots \oplus J_s$, where J_i, for $i = 1, \ldots, s$, is a Jordan block of size k_i corresponding to an eigenvalue λ_i (the λ_i's do not need to be distinct and $k_1 + \cdots + k_s = n$).

We set $f(A) = M(f(J_1) \oplus \cdots \oplus f(J_s))M^{-1}$, where

$$f(J_i) = \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \cdots & \frac{f^{(k_i-1)}(\lambda_i)}{(k_i-1)!} \\ & f(\lambda_i) & \ddots & \vdots \\ & & \ddots & f'(\lambda_i) \\ 0 & & & f(\lambda_i) \end{bmatrix},$$

for $i = 1, \ldots, s$. The definition makes sense if f is differentiable on λ_i up to the order $k_i - 1$, for $i = 1, \ldots, s$.

The principal matrix square root and the matrix sign are matrix functions, defined by $f(z) = z^{1/2}$ and $f(z) = \mathrm{sign}(z)$, respectively. With an abuse of notation, we will use the term *matrix function* also for the unitary polar factor and for the matrix geometric mean.

We recall a useful theorem which is part of a general result on the relationship between the Jordan canonical form of A and the one of $f(A)$, whose complete statement and proof can be found for instance in [10] or [14, Thm. 9.4.7]

Theorem 1. *Let $A \in \mathbb{C}^{n\times n}$, let f be such that $f(A)$ is well defined. Let λ be an eigenvalue of A such that the Jordan canonical form of A has a nontrivial Jordan block $J(\lambda)$ of size k, corresponding to λ. If $f'(\lambda) \neq 0$, then the Jordan canonical form of $f(A)$ has a Jordan block $J(f(\lambda))$ of size k.*

From the above theorem, we deduce that if f is differentiable at λ and $f'(\lambda) \neq 0$, then λ is a semisimple eigenvalue of A if and only if $f(\lambda)$ is a semisimple eigenvalue of $f(A)$.

Lemma 2. *Let A be an $n \times n$ complex matrix. Then matrix $I - 4A^2$ admits a principal square root if and only if A does not have real eigenvalues of modulus greater than $1/2$ and the real eigenvalues of modulus $1/2$ (if any) are semisimple.*

Proof. The matrix $W = I - 4A^2$ admits a principal square root if and only if W does not have negative real eigenvalues and the null eigenvalues (if any) are semisimple. The eigenvalues of W are the image under the function $f(z) = 1 - 4z^2$ of the eigenvalues of A. Therefore, the matrix W does not have negative eigenvalues if and only if the real eigenvalues (if any) of A belong to the interval $(-1/2, 1/2)$; the matrix W has a null eigenvalue if and only if A has a real eigenvalue of modulus $1/2$. In that case, since $f'(z) = 0$ if and only if $z = 0$, by

Theorem 1, the algebraic and geometric multiplicity of any nonzero eigenvalue λ of A is the same as the algebraic and geometric multiplicity of $f(\lambda)$ in $f(A)$, respectively. Therefore the two conditions are equivalent. $\square$

We recall some definitions and properties on matrix polynomials; we refer the reader to the book [5] for a complete treatise on matrix polynomials. A $n \times n$ *matrix polynomial of degree* ℓ is a polynomial $L(z) = \sum_{j=0}^{\ell} A_j z^j$, where the coefficients A_i, $i = 0, \ldots, \ell$, are $n \times n$ matrices and A_ℓ is different from the null matrix. The matrix polynomial $L(z)$ is said *regular* is $\det(L(z))$ does not vanish identically. A scalar μ is called *root* of $L(z)$ if $\det(L(\mu)) = 0$. We say that $L(z)$ has a *root at infinity* if $\mu = 0$ is a root of $\operatorname{rev}(L(z)) = \sum_{j=0}^{\ell} A_{\ell-j} z^j$. An $n \times n$ regular matrix polynomial of degree ℓ has exactly $n\ell$ roots, including the roots at infinity.

3 Quadratic palindromic matrix polynomials

In this section we study the properties of the quadratic matrix polynomial (QPMP)

$$\varphi(z) = P + Qz + Pz^2, \tag{2}$$

where P and Q are $n \times n$ complex matrices, P is different from the null matrix, and Q is nonsingular. Sometimes it will be useful to analyze the properties of the QPMP

$$\widetilde{\varphi}(z) = M + Iz + Mz^2,$$

where $M = Q^{-1}P$, which is obtained by multiplying $\varphi(z)$ on the left by Q^{-1}.

3.1 Roots of the matrix polynomial

Observe that the function $\det(Mz^{-1} + I + Mz)$ can have only a finite number of roots of moduls one. Therefore the QPMP (2) is a regular matrix polynomial.

Observe also that, if $\mu \notin \{0, \infty\}$ is a root of $\varphi(z)$, and if u is a non-zero vector such that $\varphi(\mu)u = 0$, then $\varphi(\mu^{-1})u = 0$. Therefore the roots of $\varphi(z)$, different from 0 and ∞, come in pairs (μ, μ^{-1}). If we adopt the convention that $1/0 = \infty$ and $1/\infty = 0$, then all the roots of $\varphi(z)$ come in pairs (μ, μ^{-1}). In particular if $\det(\varphi(z)) \neq 0$ on the unit circle, then $\varphi(z)$ has exactly n roots $\mu_1, \ldots, \mu_n$ in the open unit disk, and n roots outside the closed unit disk, namely $\mu_1^{-1}, \ldots, \mu_n^{-1}$.

We can give more precise results on the roots of $\varphi(z)$ by considering the Jordan canonical form of the matrix $M = Q^{-1}P$: let

$$K^{-1}MK = J_1 \oplus J_2 \oplus \cdots \oplus J_s, \tag{3}$$

where J_i is a Jordan block of size k_i corresponding to the eigenvalue λ_i. By applying the similarity transformation of (3) to $\widetilde{\varphi}(z)$, we find that the roots of $\varphi(z)$ are the union of the roots of the PQMPs $\varphi_i(z) = J_i z^2 + Iz + J_i$, for $i = 1, \ldots, s$. On the other hand, since the matrices J_i are Jordan blocks of size k_i, one has

$$\det(\varphi_i(z)) = (\lambda_i z^2 + z + \lambda_i)^{k_i}. \tag{4}$$

Therefore we recall the properties of the solutions z_1 and z_2 of the scalar palindromic quadratic equation

$$\lambda z^2 + z + \lambda = 0, \tag{5}$$

where $\lambda \in \mathbb{C}$, we assume $|z_1| \leqslant |z_2|$, and where we set $z_1 = 0$, $z_2 = \infty$ if $\lambda = 0$.

Lemma 3. *Equation* (5) *has two distinct solutions if and only if* $\lambda \neq \pm 1/2$. *Moreover,*

- *if* $\lambda \in (-\infty, -1/2] \cup [1/2, +\infty)$ *then* $|z_1| = |z_2| = 1$*;*
- *if* $\lambda \in (\mathbb{C} \setminus \mathbb{R}) \cup (-1/2, 1/2)$ *then* $|z_1| < 1 < |z_2|$.

Proof. If $\lambda \in (-\infty, -1/2] \cup [1/2, +\infty)$ then we may easily verify that $z_{1,2} = \frac{-1 \pm \mathbf{i}\sqrt{4\lambda^2-1}}{2\lambda}$, where $\mathbf{i}$ is the imaginary unit, have modulus equal to 1. If $\lambda \in (\mathbb{C} \setminus \mathbb{R}) \cup (-1/2, 0) \cup (0, 1/2)$ then $z_1 + z_2 = 1/\lambda \notin [-2, 2]$. From the equality $z_1 z_2 = 1$ we deduce that $|z_1| \leqslant 1 \leqslant |z_2|$. If $|z_1| = 1$ then $z_2 = \overline{z_1}$, hence $|z_2| = 1$ and $z_1 + z_2 \in [-2, 2]$; the latter property is false for the hypotheses on λ, therefore $|z_1| < 1$, and hence $|z_2| > 1$. $\square$

From the above lemma and from (4) we conclude that to each eigenvalue λ_i of M, with associated a Jordan block of size k_i, there correspond:

1. k_i roots of $\varphi(z)$ inside the open unit disk and k_i roots of $\varphi(z)$ outside the closed unit disk, if $\lambda_i \in (\mathbb{C} \setminus \mathbb{R}) \cup (-1/2, 1/2)$;
2. $2k_i$ roots of $\varphi(z)$ of modulus 1 if $\lambda_i \in (-\infty, -1/2] \cup [1/2, +\infty)$.

In particular, if M does not have real eigenvalues of modulus greater than or equal to 1/2, then the QPMP $\varphi(z)$ has n roots in the open unit disk, and n roots outside the closed unit disk.

3.2 Solutions of the matrix equation

We associate with $\varphi(z)$ the palindromic quadratic matrix equation

$$PX^2 + QX + P = 0, \tag{6}$$

where the unknown X is an $n \times n$ matrix. Sometimes, it will be useful to consider the equivalent matrix equation

$$MX^2 + X + M = 0, \tag{7}$$

where $M = Q^{-1}P$.

Let X be a solution and let u be a non-zero vector such that $Xu = \mu u$. Then, by multiplying (6) on the right by the vector u, we get

$$(P\mu^2 + Q\mu + P)u = 0,$$

i.e., μ is a root of $\varphi(z)$, and u is a vector in the kernel of $\varphi(\mu)$. Therefore the eigenvalues of any solution X are a subset of the roots of $\varphi(z)$.

We give conditions for the existence and uniqueness of a solution X having spectral radius at most 1, which is a function of M. The case where M is a Jordan block is treated by the following

Theorem 4. *Let J be a Jordan block of dimension k, associated with the eigenvalue λ, and consider the matrix equation*

$$JX^2 + X + J = 0. \tag{8}$$

The following properties hold:

1. *If $\lambda \in (\mathbb{C} \setminus \mathbb{R}) \cup (-1/2, 1/2)$, or if $\lambda = \pm 1/2$ and $k = 1$, then $X = -2J(I + (I - 4J^2)^{1/2})^{-1}$ is the unique solution of* (8) *which is a function of J, having spectral radius at most one.*

2. *If $\lambda = \pm 1/2$ and $k > 1$ then equation* (8) *does not have a solution which is a function of J, having spectral radius at most one.*

3. *If $\lambda \in (-\infty, -1/2) \cup (1/2, +\infty)$ then equation* (8) *has more than one solution which is a function of J, having spectral radius one.*

Proof. If $\lambda \in (\mathbb{C} \setminus \mathbb{R}) \cup (-1/2, 1/2)$, or if $\lambda = \pm 1/2$ and $k = 1$, then the matrix $X = -2J(I + (I - 4J^2)^{1/2})^{-1}$ is well defined by virtue of Lemma 2 and is a function of J. By direct inspection one may verify that X is a solution of (8). Moreover its eigenvalues are the image under $z \to -2z/(1 + \sqrt{1 - 2z^2})$ of the eigenvalues of J; therefore the spectral radius of X is at most one.

If $\lambda \in (\mathbb{C} \setminus \mathbb{R}) \cup (-1/2, 1/2)$ then the k eigenvalues of X are the roots in the open unit disk of $\varphi(z) = Jz^2 + Iz + J$. Therefore X is the unique solution with spectral radius less than one (see for instance Section 3.3 of [5], or Theorem 3.18 of [2]).

If $\lambda = 1/2$ and $k = 1$, then the unique solution of (8) is $X = -1$. If $\lambda = 1/2$ and J has size $k > 1$, we show that equation (8) does not have solution. Assume that X is a solution. Since the eigenvalues of X are a subset of the roots of $\varphi(z)$, which are all equal to -1, it follows that the only eigenvalue of X is -1. Therefore the eigenvalues of the matrix $Z = (I + X)(I - X)^{-1}$ are all equal to 0; hence $Z^m = 0$ for any $m \geqslant k$. On the other hand, by recovering $I + 2X$ and $I - 2X$ from (8), one finds that $Z^2 = (I - 2J)(I + 2J)^{-1}$. By direct inspection, one observes that $\left((I - 2J)(I + 2J)^{-1}\right)^p \neq 0$ for any $p < k$. Therefore $Z^{2(k-1)} \neq 0$, that contradicts the fact that $Z^m = 0$ for any $m \geqslant k$.

The case $\lambda = -1/2$ is treated analogously.

If $\lambda \in (-\infty, -1/2) \cup (1/2, +\infty)$ then, by Lemma 3 and (4), $\varphi(z)$ has k roots equal to z_1 and k roots equal to z_2, where $z_1 \neq z_2$ and $|z_1| = |z_2| = 1$. The matrices $Y_\pm = -\frac{1}{2}J^{-1}(I \pm \mathbf{i}(4J^2 - I)^{1/2})$ are functions of J, and solve equation (8). The unique eigenvalue of Y_+ and Y_- is z_1 and z_2, respectively. Therefore $Y_+ \neq Y_-$ and they both have spectral radius 1, so the solution is not unique. □

An immediate consequence of the above result is the next theorem, which shows that just the location of the eigenvalues of $M = Q^{-1}P$ gives necessary and sufficient conditions for the existence of a unique solution X_* of (6), which is a function of M and with spectral radius at most one.

Theorem 5. *Let P, Q be two square matrices, with Q nonsingular, and set $M = Q^{-1}P$. Then the following conditions are equivalent:*

1. *the matrix M does not have real eigenvalues of modulus greater than $1/2$ and the real eigenvalues of modulus $1/2$ (if any) are semisimple;*

2. *the matrix equation* (6) *has a unique solution X_* which is a function of M and whose eigenvalues lie in the closed unit disk; moreover, its explicit expression is*

$$X_* = -2M(I + (I - 4M^2)^{1/2})^{-1}. \tag{9}$$

Proof. Consider the Jordan decomposition of M, given in (3). Any solution X which is a function of M is such that $K^{-1}XK$ is block diagonal, with the same block structure as $K^{-1}MK$ (see [9]). Therefore, by applying the similarity transformation defined by K to equation (7), one obtains s uncoupled matrix equations

$$J_iY^2 + Y + J_i = 0, \tag{10}$$

where Y has size k_i. Equation (7) has a unique solution which is a function of M and whose eigenvalues lie in the closed unit disk if and only if each of the equations (10) has a unique solution Y_i which is a function of J_i (the same function for each J_i) and whose eigenvalues lie in the closed unit disk. In that case the unique solution of (7) is $X_* = K(Y_1 \oplus \cdots \oplus Y_s)K^{-1}$. The proof is completed by applying Theorem 4 at each of the matrix equations (10). □

Theorem 5 implies that, if condition 1 is satisfied, then X_* is the unique solution having spectral radius at most one, which is a function of M. Since the eigenvalues of any solution X of (6) are a subset of the roots of $\varphi(z)$, one deduces that X_* is a solution of smallest spectral radius, and it is the unique which is a function of M. In fact, there might exist other solutions having spectral radius at most one, which are not function of M, as shown by the following example.

Example 6. *Consider the matrix equation*

$$X^2 + 2X + I = 0. \tag{11}$$

For this equation $M = \frac{1}{2}I$, therefore the only eigenvalue of the matrix M is $1/2$, which is semisimple. According to Theorem 5, $X_* = -I$ is the unique solution of spectral radius at most 1, which is a function of M. On the other hand, if $n = 2$, for any scalars x, y, with $y \neq 0$, the matrix

$$X = \begin{bmatrix} x & y \\ -\frac{x^2+2x+1}{y} & -2-x \end{bmatrix}$$

is a solution of (11) *having spectral radius 1. Therefore equation* (11) *has infinite solutions having spectral radius one, which are not function of M.*

If the matrix M does not have real eigenvalues of modulus greater than or equal to $1/2$, then X_* is the unique solution having spectral radius less than one (see for instance [2, Theorem 3.18]) .

3.3 Laurent matrix polynomials

In this section we study the invertibility domain of the palindromic Laurent matrix polynomial

$$\mathcal{L}(z) = Pz^{-1} + Q + Pz, \tag{12}$$

which is obtained by multiplying $\varphi(z)$ by z^{-1}.

Let X_* be defined in (9). Since X_* commutes with $M = Q^{-1}P$, one may easily verify that the following factorization holds

$$Mz^{-1} + I + Mz = (I - X_*z)(I + MX_*)(I - X_*z^{-1}), \quad z \neq 0,$$

which implies

$$\mathcal{L}(z) = (I - QX_*Q^{-1}z)(Q + PX_*)(I - X_*z^{-1}), \quad z \neq 0. \tag{13}$$

Lemma 7. *The Laurent matrix polynomial* (12) *is invertible in an open annulus containing the unit circle if and only if the matrix $M = Q^{-1}P$ does not have real eigenvalues of modulus greater than or equal to* $1/2$. *In that case the invertibility domain of $\mathcal{L}(z)$ is the annulus $\mathcal{A}_R = \{R < |z| < 1/R\}$, where $R = \rho(X_*)$, and X_* is defined in* (9). *Moreover, by setting $\mathcal{L}(z)^{-1} = \mathcal{H}(z) = \sum_{i=-\infty}^{+\infty} H_i z^i$ one has $H_i = H_{-i}$ for $i > 0$ and*

$$H_0 = (I - 4M^2)^{-1/2} Q^{-1}. \tag{14}$$

Proof. If M does not have real eigenvalues of modulus greater than or equal to $1/2$, then from Theorem 5 and from the factorization (13) it follows that the roots of $\det \mathcal{L}(z)$ are the eigenvalues of X_* and their reciprocals. Therefore, since $\rho(X_*) < 1$, $\mathcal{L}(z)$ is invertible in the annulus $\mathcal{A}_R$, where $R = \rho(X_*)$. Conversely, if M has a real eigenvalue outside the interval $(-1/2, 1/2)$, from Lemma 3 the polynomial $\lambda z^2 + z + \lambda$ has at least one root of modulus one, therefore the function $\det \mathcal{L}(z)$ has some roots on the unit circle, thus, $\mathcal{L}(z)$ cannot be invertible on the unit circle. Since $\mathcal{L}(z) = \mathcal{L}(z^{-1})$, one deduces that $\mathcal{H}(z) = \mathcal{H}(z^{-1})$, therefore $H_i = H_{-i}$ for $i > 0$. From (13) one has

$$H_0 = \sum_{i=0}^{\infty} X_*^i (Q + PX_*)^{-1} Q X_*^i Q^{-1} = \sum_{i=0}^{\infty} X_*^i (I + MX_*)^{-1} X_*^i Q^{-1}.$$

Since M and X_* commute, being X_* a function of M (see [9, Thm. 1.13]), and since $\rho(X_*) < 1$, from the above equality one obtains that

$$\begin{aligned} H_0 &= (I - X_*^2)^{-1}(I + MX_*)^{-1} Q^{-1} = \left((I - X_*^2)(I + MX_*)\right)^{-1} Q^{-1} \\ &= (I + MX_* - (X_* + MX_*^2)X_*)Q^{-1} = (I + 2MX_*)^{-1} Q^{-1}. \end{aligned}$$

Replacing X_* with (9) yields (14). $\square$

4 Relationships with matrix functions

By extending a result of [16] on the matrix square root, it is possible to give a functional interpretation of the matrix sign, the polar factor of a nonsingular matrix and the matrix geometric mean of two positive matrices, in terms of the quadratic palindromic matrix polynomial (2). If we write $P = \frac{1}{4}(S - T)$, $Q = \frac{1}{2}(S + T)$, according to the properties of the matrices S and T, the constant coefficient H_0 of $\mathcal{L}(z)^{-1}$ provides the different matrix functions. To better understand this fact, it is useful to introduce the notation

$$\mathcal{L}(z; S, T) = \frac{1}{4}(S - T)z^{-1} + \frac{1}{2}(S + T) + \frac{1}{4}(S - T)z.$$

and

$$\xi(V) = \rho((V - I)(V + I)^{-1}),$$

where V is any matrix such that $V + I$ is nonsingular, and $\rho(\cdot)$ denotes the spectral radius.

The following theorem gives a functional interpretation of $A^{1/2}$, $\mathrm{sign}(A)$, $\mathrm{polar}(A)$, $A\#B$, in terms of the constant coefficient H_0 of the Laurent matrix polynomial $\mathcal{L}(z; S, T)^{-1}$.

Theorem 8. *The following properties hold:*

1. *If A is a matrix having no nonpositive real eigenvalues, $S = I$ and $T = A^{-1}$, then $\mathcal{L}(z; S, T)$ is invertible in $\mathcal{A}_R$, where $R = \xi(A^{-1/2})$, and $H_0 = A^{1/2}$.*

2. *If A is a matrix having no imaginary eigenvalues, $S = A^{-1}$ and $T = A$, then $\mathcal{L}(z; S, T)$ is invertible in $\mathcal{A}_R$, where $R = \xi((A^2)^{1/2})$, and $H_0 = \operatorname{sign}(A)$.*

3. *If A is nonsingular, $S = A^{-1}$, $T = A^*$, then $\mathcal{L}(z; S, T)$ is invertible in $\mathcal{A}_R$, where $R = \xi((AA^*)^{1/2})$, and $H_0 = \operatorname{polar}(A)$.*

4. *If A and B are positive definite matrices, $S = A^{-1}$ and $T = B^{-1}$, then $\mathcal{L}(z; S, T)$ is invertible in $\mathcal{A}_R$, where $R = \xi((B^{-1}A)^{1/2})$, and $H_0 = A\#B$.*

Proof. Part 1 has been proved in [16]. Part 2 follows from part 1, by observing that $A^{-1}\mathcal{L}(z; I, A^2) = \mathcal{L}(z; A^{-1}, A)$; the constant coefficient of $\mathcal{L}(z; I, A^2)^{-1}$ is $(A^{-2})^{1/2}$, thus the constant coefficient of $\mathcal{L}(z; A^{-1}, A)^{-1}$ is $A(A^{-2})^{1/2} = A(A^2)^{-1/2} = \operatorname{sign}(A)$.

Concerning part 3, as in the previous case, we observe that $\mathcal{L}(z; A^{-1}, A^*) = \mathcal{L}(z; I, A^*A)A^{-1}$. The constant coefficient of $\mathcal{L}(z; I, A^*A)^{-1}$ is $(A^*A)^{-1/2}$, thus the constant coefficient of $\mathcal{L}(z; A^{-1}, A^*)^{-1}$ is $A(A^*A)^{-1/2} = \operatorname{polar}(A)$.

Concerning part 4, since $\mathcal{L}(z; A^{-1}, B^{-1}) = \mathcal{L}(z; I, B^{-1}A)A^{-1}$, the constant coefficient of $\mathcal{L}(z; I, B^{-1}A)^{-1}$ is $(B^{-1}A)^{-1/2} = (A^{-1}B)^{1/2}$, thus the constant coefficient of $\mathcal{L}(z; A^{-1}, B^{-1})^{-1}$ is $A(A^{-1}B)^{1/2} = A\#B$. □

Table 1 synthesizes the results of Theorem 8:

H_0	S	T
$A^{1/2}$	I	A^{-1}
$\operatorname{sign}(A)$	A^{-1}	A
$\operatorname{polar}(A)$	A^{-1}	A^*
$A\#B$	A^{-1}	B^{-1}

Table 1: The coefficient H_0 of $\mathcal{L}(z; S, T)^{-1}$

5 Integral representations

We consider some integral representations for $A^{1/2}$, $\operatorname{sign}(A)$, $\operatorname{polar}(A)$, $A\#B$. One of them is obtained as a byproduct of Theorem 8. Indeed, since the function $\mathcal{H}(z) = \mathcal{L}(z)^{-1}$ is convergent in an annulus containing the unit circle, from a classic result on matrix Laurent power series belonging to the Wiener algebra (see for instance Theorem 3.1 of [2]), one obtains

$$H_0 = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}(e^{\mathbf{i}\vartheta}) d\vartheta,$$

where $\mathbf{i}$ is the imaginary unit. Observe that $\mathcal{H}(e^{\mathbf{i}\vartheta}) = (Q + 2P\cos\vartheta)^{-1}$. Therefore one has

$$H_0 = \frac{1}{2\pi} \int_0^{2\pi} (Q + 2P\cos\vartheta)^{-1}\, d\vartheta. \tag{15}$$

The latter equation, combined with Theorem 8, allows one to derive new integral representations of $A^{1/2}$, $\mathrm{sign}(A)$, $\mathrm{polar}(A)$, $A\#B$. For instance, if $P = \frac{1}{4}(A^{-1} - B^{-1})$ and $Q = \frac{1}{2}(A^{-1} + B^{-1})$, we get

$$A\#B = \frac{1}{\pi} A \left(\int_0^{2\pi} (B - A + (A + B)\cos\vartheta)^{-1}\, d\vartheta \right) B.$$

Another interesting property is that the integral representation (15) and many other integral representations which can be found in the literature can all be obtained from the Cauchy formula for the inverse square root.

In fact, if V has no nonpositive real eigenvalues, the Cauchy formula yields

$$V^{-1/2} = \frac{1}{2\pi\mathbf{i}} \int_\gamma \zeta^{-1/2}(\zeta I - V)^{-1} d\zeta,$$

where γ is the Hankel contour (compare [6, p. 458]) that goes from $-\infty$ to 0 and back.

Each of the matrix functions $A^{1/2}$, $\mathrm{sign}(A)$, $\mathrm{polar}(A)$, $A\#B$ can be expressed in the form $F = UV^{-1/2}$, for suitable matrices U and V. Therefore each matrix function has the integral representation

$$F = \frac{U}{2\pi\mathbf{i}} \int_\gamma \zeta^{-1/2}(\zeta I - V)^{-1} d\zeta, \tag{16}$$

where the matrices U and V are given in Table 2.

F	U	V
$A^{1/2}$	A	A
$\mathrm{sign}(A)$	A	A^2
$\mathrm{polar}(A)$	A	A^*A
$A\#B$	A	$A^{-1}B$

Table 2: U and V in the integral representations

Formula (16) allows one to derive and generalize specific representations well known in the literature. Indeed, the substitution $t = \mathbf{i}\zeta^{1/2}$ leads to the formula

$$F = \frac{2}{\pi} U \int_0^\infty (t^2 I + V)^{-1} dt,$$

which, for $U = A$ and $V = A^2$, yields the representation of the matrix sign given in [11].

The change of variable $\varphi = \tan(t)$ yields

$$F = \frac{2}{\pi} U \int_0^{\pi/2} (\sin^2\varphi\ I + \cos^2\varphi\ V)^{-1} d\varphi.$$

In the specific case where $F = \mathrm{sign}(A)$ we rediscover the integral representation of the matrix sign function given in [11].

The change of variable $\psi = \pi/2 - \varphi$ yields

$$F = \frac{2}{\pi} U \int_0^{\pi/2} (\cos^2\varphi\ I + \sin^2\varphi\ V)^{-1} d\varphi. \tag{17}$$

The substitution $\vartheta = 2\varphi$ and the bisection formulae for the sine and cosine lead to

$$F = \frac{2}{\pi} U \int_0^{\pi} ((V+I) + \cos\vartheta(V-I))^{-1} d\vartheta.$$

By using the symmetry of the cosine function, one obtains

$$F = \frac{U}{\pi} \int_0^{2\pi} ((V+I) + \cos\vartheta(V-I))^{-1} d\vartheta,$$

which leads to equation (15).

Another interesting representation can be obtained setting $t = \cos\varphi$ in equation (17):

$$F = \frac{1}{\pi} U \int_0^1 \frac{((1-t)I + tV)^{-1}}{\sqrt{t(1-t)}} dt.$$

In the specific case where $F = A\#B$ we obtain the integral representation of the matrix mean, proved in [1] by using an Eulerian integral.

Finally, by setting $s = 2t - 1$, one obtains the formula

$$F = \frac{2}{\pi} U \int_{-1}^1 \frac{((1-s)I + (1+s)V)^{-1}}{\sqrt{1-s^2}} ds.$$

which is well suited for the Gauss-Chebyshev quadrature.

Therefore, by starting from (16) and by performing specific changes of variables, we have seen in a unifying framework specific known integral representations of the matrix functions $A^{1/2}$, $\mathrm{sign}(A)$, $\mathrm{polar}(A)$, $A\#B$.

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